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# DEVELOPMENTS IN VARIATIONAL METHODS FOR HIGH PERFORMANCE PLATE AND SHELL ELEMENTS

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## ABSTRACT

High performance elements are simple finite elements constructed to deliver engineering accuracy with coarse arbitrary grids. This paper is part of a series on the variational foundations of high-performance elements, with emphasis on plate and shell elements constructed with the free formulation (FF) and assumed natural strain (ANS) methods. In the present paper we study parametrised variational principles that provide a common foundation for the FF and ANS methods, as well as for a combination of both. From this unified formulation a variant of the ANS formulation, called the assumed natural deviatoric strain (ANDES) formulation, emerges as an important special case. The first ANDES element: a high-performance 9-dof triangular Kirchhoff plate bending element, is briefly described to illustrate the use of the new formulation.

## 1. INTRODUCTION

For 25 years researchers have tried to construct "best" finite element models for problems in structural mechanics. The quest appeared to be nearly over in the late 1960s when high order displacement elements dominated the headlines. But these elements did not dominate the marketplace. The overwhelming preference of finite element code users has been for *simple elements that deliver engineering accuracy with coarse meshes*. These will be collectively called *high performance elements*, or HP elements for short.

### 1.1 Attributes of HP Elements

Approaching that general goal gives rise to a myriad of more concrete requirements, which are supposed to be addressed in higher or lesser degree during element development. Such requirements are listed in Table 1.

Some of these requirements are obvious. For example, *low distortion sensitivity* is a consequence of trying to achieve satisfactory accuracy with *arbitrary meshes*. But other items listed in Table 1 call for some explanation.

Table 1. Target Requirements for High-Performance Elements

- Simple: few freedoms, all physical, preferably at corners only
- Convergent
- Frame invariant
- No locking
- Rank sufficient: no spurious modes
- Balanced stiffness: not too rigid, not too flexible
- Stresses as accurate as displacements
- Low distortion sensitivity
- Mixable with other elements
- Economical to form
- Easily extendible to nonlinear and dynamic analysis
- Effective local error estimator for mesh adaptation

The first and foremost requirement is that the element be as *simple* as possible. This is in sharp contrast to the "baroque FE period" of 1965-1975 that lauded luxuriantly ornate elements and culminated with the development of very complex models, including elements with nonphysical degrees of freedom. One source of this retrenchment has been feedback from users of general-purpose finite element programs. As use of these programs expands to more engineers without deep knowledge of "what's inside the black box" the trend in finite element model construction has veered towards the "simplest elements that will do the job." Further impetus is provided by the gradual realization that high accuracy of complex elements in linear elastostatics does not necessarily carry over to dynamic and nonlinear analysis.

The *balanced stiffness* requirement also deserves comment. It follows from the goal of attaining reasonable accuracy with *coarse meshes*. This is illustrated in Fig. 1, which shows a convergence study of a classical model problem: the bending of a simply-supported square plate under a concentrated central load. The mesh contains  $2 \times N \times N$  triangles over a plate quadrant. A target "accuracy band" of  $\pm 1\%$  is taken, somewhat arbitrarily, as representative of engineering accuracy for this rather simple problem. The convergence characteristics of several triangular elements are taken from the extensive study reported in Ref. 2. Although most elements converge, some are too stiff while others are too flexible, and generally do not enter the accuracy band until the mesh is fairly refined ( $N \geq 8$ ). On the other hand, the results labeled 'FF', obtained with a plate element based on the free formulation (FF) discussed later, lie within the band for all meshes.

The balanced-stiffness requirement should *not* be confused with fast asymptotic convergence for *fine* meshes. Simple elements cannot effectively compete with higher order elements in this regard, and are not effective in applications that demand very high accuracy. What is important is *how good are the results for coarse meshes*.

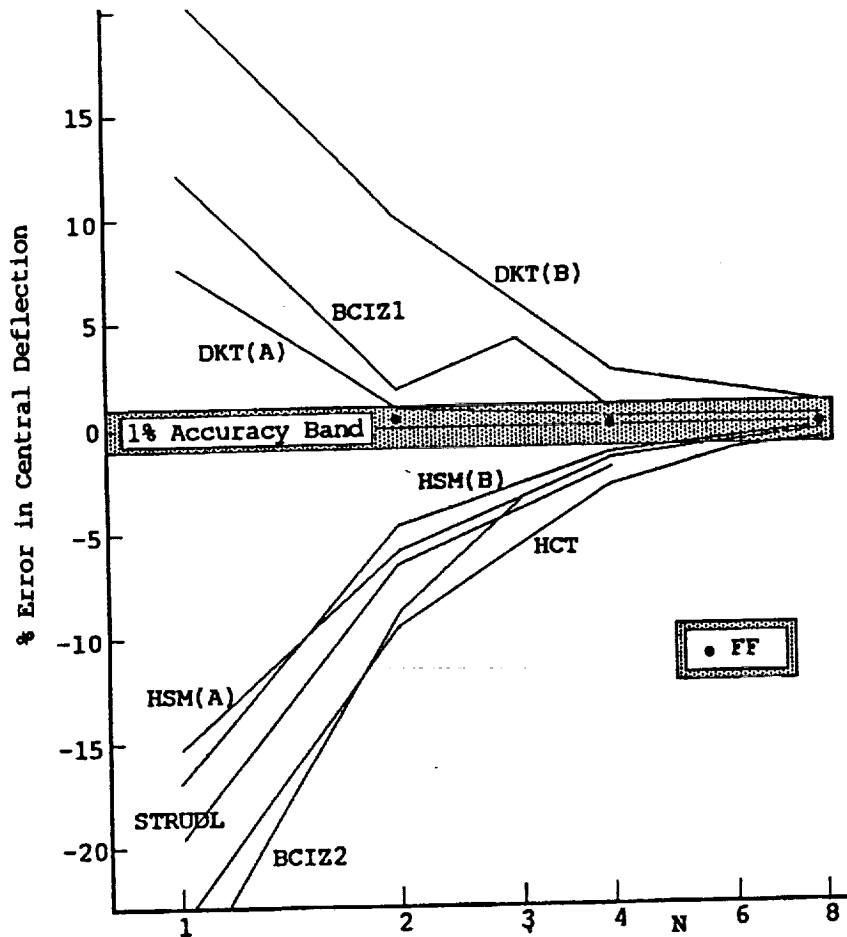


Figure 1. Convergence study of several plate bending triangular elements as reported in Ref. 2. The FF results are from Ref. 8.

### 1.2 Constructing HP Elements

The search for high-performance (HP) elements began seriously in the mid 1970s and by now it represents an important area of finite element research in solid and structural mechanics. Many ingenious schemes have been tried: reduced and selective integration, incompatible modes, mixed and hybrid formulations, stress and strain projections, the free formulation (FF), and the assumed natural strain (ANS) formulation. Many researchers are presently working to develop such elements. The common theme of the investigations is

*Abandon the conventional displacement formulation*

Several techniques used by researchers in their quest to build better elements are itemised in Table 2. It may be noted that many of these were introduced over 20 years ago. But it is only recently that a concerted effort is made to *combine* several tools to produce HP elements. For example, the present work draws on items 1, 2, 3, 8, 10, 11 and 12 of Table 2.

Table 2. Tools of the Trade

	<i>Technique</i>	<i>Year introduced</i>
1.	Incompatible shape functions	early 1960s
2.	Patch test	1965
3.	Mixed and hybrid variational principles	1965
4.	Projectors	1967
5.	Selective reduced integration	1969
6.	Uniform reduced integration	1970
7.	Assumed strains	1970
8.	Energy balancing	1974
9.	Directional integration	1978
10.	Limit differential equations	1982
11.	Free formulation	1984
12.	Assumed natural strains	1984

### 1.3 Objective of Present Work

This paper is part of a series (Refs. 9-12, 15-16) that studies how several HP element construction methods can be embedded within an *extended* variational framework that uses parametrised hybrid functionals. Particular attention is focused on merging the last two items in Table 2.

The general plan of attack for this unification is flowcharted in Fig. 2. Box connections indicated with dashed lines are not dealt with in the present work. The variational extensions, shown on the left of Fig. 2, involve parametrisation of the conventional elasticity functionals and treatment of element interfaces through generalizations of the hybrid approach of Refs. 20-23.

The effective construction of HP elements relies on devices, sometimes derisively called "tricks" or "variational crimes," that do not fit *a priori* in the classical variational framework. The range of tricks spans innocuous collocation and finite difference constraints to more drastic remedies such as selective integration. Despite their unconventional nature, tricks are an essential part of the construction of high-performance elements. Collectively they represent a fun-and-games ingredient that keeps the derivation of HP finite elements a surprisingly enjoyable task.

The present treatment "decriminalises" kinematic constraint tricks by adjoining Lagrange multipliers, hence setting out the ensemble on proper variational foundations. Placing formulations within a variational framework has the great advantage of supplying the *general structure* of the stiffness matrices and forcing vectors of high performance elements, and of providing theoretical coherence for the systematic derivation of classes of elements by a combination of techniques.

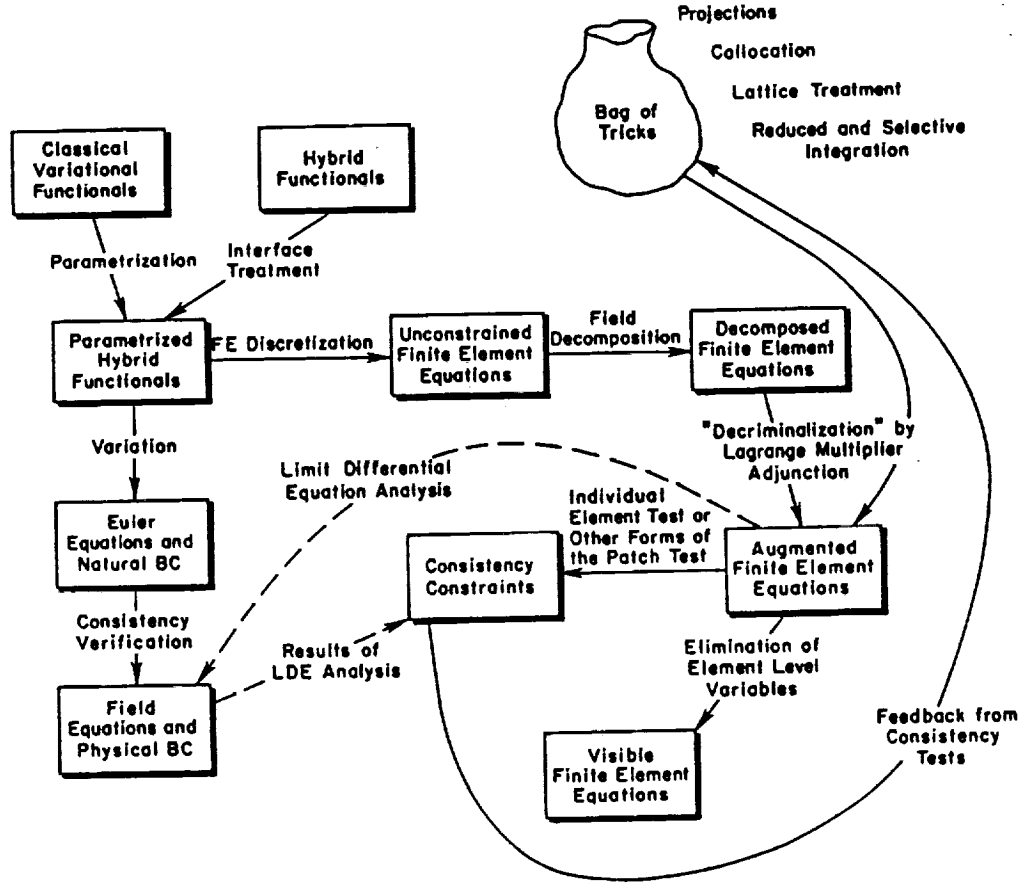


Figure 2 Program of attack for variational formulation of HP elements

## 2. THE ELASTICITY PROBLEM

Consider a *linearly elastic body* under static loading that occupies the volume  $V$ . The body is bounded by the surface  $S$ , which is decomposed into  $S : S_d \cup S_t$ . Displacements are prescribed on  $S_d$  whereas surface tractions are prescribed on  $S_t$ . The outward unit normal on  $S$  is denoted by  $\mathbf{n} \equiv \mathbf{n}_i$ .

The three unknown volume fields are displacements  $\mathbf{u} \equiv u_i$ , infinitesimal strains  $\mathbf{e} \equiv e_{ij}$ , and stresses  $\boldsymbol{\sigma} \equiv \sigma_{ij}$ . The problem data include: the body force field  $\mathbf{b} \equiv b_i$  in  $V$ , prescribed displacements  $\hat{\mathbf{d}} \equiv \hat{d}_i$  on  $S_d$ , and prescribed surface tractions  $\hat{\mathbf{t}} \equiv \hat{t}_i$  on  $S_t$ .

The relations between the volume fields are the strain-displacement equations

$$\mathbf{e} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}) = \mathbf{D} \mathbf{u} \quad \text{or} \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V, \quad (1)$$

the constitutive equations

$$\boldsymbol{\sigma} = \mathbf{E} \mathbf{e} \quad \text{or} \quad \sigma_{ij} = E_{ijkl} e_{kl} \quad \text{in } V, \quad (2)$$

which will be assumed to be invertible, and the equilibrium (balance) equations

$$-\text{div } \boldsymbol{\sigma} = \mathbf{D}^* \boldsymbol{\sigma} = \mathbf{b} \quad \text{or} \quad \sigma_{ij,j} + b_i = 0 \quad \text{in } V, \quad (3)$$

in which  $D^* = -\text{div}$  denotes the adjoint operator of  $D = \frac{1}{2}(\nabla + \nabla^T)$ .

The stress vector with respect to a direction defined by the unit vector  $\mathbf{v}$  is denoted as  $\sigma_v = \sigma \cdot \mathbf{v}$ , or  $\sigma_{vi} = \sigma_{ij}v_j$ . On  $S$  the surface-traction stress vector is defined as

$$\sigma_n = \sigma \cdot \mathbf{n}, \quad \text{or} \quad \sigma_{ni} = \sigma_{ij}n_j. \quad (4)$$

With this definition the traction boundary conditions may be stated as

$$\sigma_n = \hat{t} \quad \text{or} \quad \sigma_{ij}n_j = \hat{t}_i \quad \text{on } S_t, \quad (5)$$

and the displacement boundary conditions as

$$\mathbf{u} = \hat{\mathbf{d}} \quad \text{or} \quad u_i = \hat{d}_i \quad \text{on } S_d. \quad (6)$$

### 3. NOTATION

#### 3.1 Field Dependency

In variational methods of approximation we do not work of course with the exact fields that satisfy the governing Eqs. 1-3 and 5-6, but with *independent* (primary) fields, which are subject to variations, and *dependent* (secondary, associated, derived) fields, which are not. The approximation is determined by taking variations with respect to the independent fields.

Following the notation introduced in Refs. 9 and 10, an *independently varied* field will be identified by a superposed tilde, for example  $\tilde{\mathbf{u}}$ . A dependent field is identified by writing the independent field symbol as superscript. For example, if the displacements are independently varied, the derived strain and stress fields are

$$\mathbf{e}^* = \frac{1}{2}(\nabla + \nabla^T)\tilde{\mathbf{u}} = D\tilde{\mathbf{u}}, \quad \sigma^* = E\mathbf{e}^* = ED\tilde{\mathbf{u}}. \quad (7)$$

An advantage of this convention is that  $\mathbf{u}$ ,  $\mathbf{e}$  and  $\sigma$  may be reserved for the exact fields.

#### 3.2 Integral Abbreviations

Volume and surface integrals will be abbreviated by placing domain-subscripted parentheses and square brackets, respectively, around the integrand. For example:

$$(f)_V \stackrel{\text{def}}{=} \int_V f dV, \quad [f]_S \stackrel{\text{def}}{=} \int_S f dS, \quad [f]_{S_d} \stackrel{\text{def}}{=} \int_{S_d} f dS, \quad [f]_{S_t} \stackrel{\text{def}}{=} \int_{S_t} f dS. \quad (8)$$

If  $\mathbf{f}$  and  $\mathbf{g}$  are vector functions, and  $\mathbf{p}$  and  $\mathbf{q}$  tensor functions, their inner product over  $V$  is denoted in the usual manner

$$(\mathbf{f}, \mathbf{g})_V \stackrel{\text{def}}{=} \int_V \mathbf{f} \cdot \mathbf{g} dV = \int_V f_i g_i dV, \quad (\mathbf{p}, \mathbf{q})_V \stackrel{\text{def}}{=} \int_V \mathbf{p} : \mathbf{q} dV = \int_V p_{ij} q_{ij} dV, \quad (9)$$

and similarly for surface integrals, in which case square brackets are used.

#### 3.3 Domain Assertions

The notation

$$(\mathbf{a} = \mathbf{b})_V, \quad [\mathbf{a} = \mathbf{b}]_S, \quad [\mathbf{a} = \mathbf{b}]_{S_d}, \quad [\mathbf{a} = \mathbf{b}]_{S_t}, \quad (10)$$

is used to assert that the relation  $\mathbf{a} = \mathbf{b}$  is valid at each point of  $V$ ,  $S$ ,  $S_d$  and  $S_t$ , respectively.

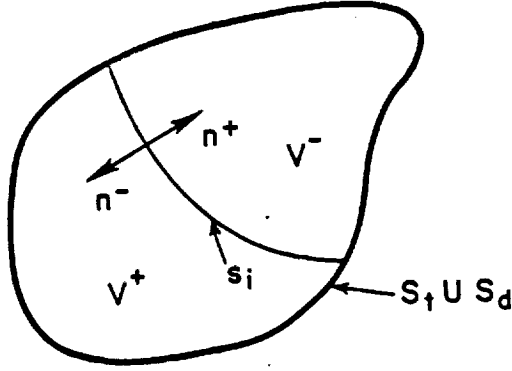


Figure 3. Internal interface example.

### 3.4 Internal Interfaces

In §4-5 we construct *hybrid variational principles* in which boundary displacements  $d$  can be varied independently from the internal displacements  $u$ . These displacements play the role of Lagrange multipliers that relax internal displacement continuity. Variational principles containing  $d$  will be called *displacement-generalized*, or *d-generalized* for short.

The choice of  $d$  as independent field is not variationally admissible on  $S_d$  or  $S_t$ . We must therefore extend the definition of boundary to include *internal interfaces* collectively designated as  $S_i$ . Thus

$$S : S_d \cup S_t \cup S_i. \quad (11)$$

On  $S_i$  neither displacements nor tractions are prescribed. A simple case is illustrated in Fig. 3, in which the interface  $S_i$  divides  $V$  into two subvolumes:  $V^+$  and  $V^-$ . An interface such as  $S_i$  on Fig. 3 has two "sides" called  $S_i^+$  and  $S_i^-$ , which identify  $S_i$  viewed as boundary of  $V^+$  and  $V^-$ , respectively. At smooth points of  $S_i$  the unit normals  $n^+$  and  $n^-$  point in opposite directions.

The integral abbreviations of Eqs. 8-9 generalize as follows, using Fig. 3 for definiteness. A volume integral is the sum of integrals over the subvolumes:

$$(f)_V \stackrel{\text{def}}{=} \int_{V^+} f dV + \int_{V^-} f dV. \quad (12)$$

An integral over  $S_i$  includes two contributions:

$$[g]_{S_i} \stackrel{\text{def}}{=} \int_{S_i^+} g^+ dS + \int_{S_i^-} g^- dS, \quad (13)$$

where  $g^+$  and  $g^-$  denotes the value of the integrand  $g$  on  $S_i^+$  and  $S_i^-$ , respectively. These two values may be different if  $g$  is discontinuous or involves a projection on the normals.

The appearance of  $S_i$  is a natural consequence of use of finite elements with discontinuous displacements. Following a finite element discretisation, the union of interelement

boundaries becomes  $S_i$ . This boundary is generally nonphysical because it depends on the discretisation.<sup>1</sup>

#### 4. THE ELASTICITY FUNCTIONALS

The variational principles of linear elasticity are based on functionals of the form

$$\Pi = U - P, \quad (14)$$

where  $U$  characterises the internal energy stored in the body volume and  $P$  includes other contributions such as work of applied loads and energy stored on internal interfaces. We shall call  $U$  the *generalized strain energy* and  $P$  the *forcing potential*.

The functionals considered in this section include *independently varied displacements*. The class of dual functionals such as the complementary energy are briefly covered in §5.5 for completeness, but are not required in the finite element developments of §6ff.

##### 4.1 Generalized Strain Energy

The generalized strain energy has the following structure:

$$U = \frac{1}{2}j_{11}(\bar{\sigma}, e^e)_V + j_{12}(\bar{\sigma}, \bar{e})_V + j_{13}(\bar{\sigma}, e^u)_V + \frac{1}{2}j_{22}(\sigma^e, \bar{e})_V + j_{23}(\sigma^e, e^u)_V + \frac{1}{2}j_{33}(\sigma^u, e^u)_V \quad (15)$$

where  $j_{11}$  through  $j_{33}$  are numerical coefficients. For example, the Hu-Washisu principle is obtained by setting  $j_{12} = -1$ ,  $j_{13} = 1$ ,  $j_{22} = 1$ , all others being zero. The matrix representation of the general functional Eq. 15 and the relations that must exist between the coefficients are studied in §5.1.

##### 4.2 Hybrid Forcing Potentials

Variational principles of linear elasticity are constructed by combining the volume integral of Eq. 15 with the forcing potential  $P$ . Two forms of the forcing potential, called  $P^d$  and  $P^t$  in the sequel, are of interest in the hybrid treatment of interface discontinuities. The  $d$ -generalized (displacement-generalized) forcing potential introduces, as anticipated in §3.4, an independent boundary displacement field  $\bar{d}$  over  $S_i$ :

$$P^d(\bar{u}, \bar{\sigma}, \bar{d}) = (b, \bar{u})_V + [\bar{\sigma}_n, \bar{u} - \bar{d}]_{S_i} + [\bar{t}, \bar{u}]_{S_i} + [\bar{\sigma}_n, \bar{u} - \bar{d}]_{S_i}. \quad (16)$$

The  $t$ -generalized (traction generalized) forcing potential introduces an independently varied traction displacement field  $\bar{t}$  over  $S_i$ :

$$P^t(\bar{u}, \bar{\sigma}, \bar{t}) = (b, \bar{u})_V + [\bar{t}, \bar{u} - \bar{d}]_{S_i} + [\bar{t}, \bar{u}]_{S_i} + [\bar{t}, \bar{u}]_{S_i}. \quad (17)$$

The "conventional" form  $P^c$  of the forcing potential is obtained if the interface integral vanishes and one sets  $[t = \sigma_n]_{S_i}$ . If so  $P^t$  and  $P^d$  coalesce into  $P^c$ , which retains only two independent fields:

$$P^c(\bar{u}, \bar{\sigma}) = (b, \bar{u})_V + [\bar{\sigma}_n, \bar{u} - \bar{d}]_{S_i} + [\bar{t}, \bar{u}]_{S_i}. \quad (18)$$

<sup>1</sup> If there are physical internal interfaces — for example a sudden thickness or material change — it is common practice to select the mesh so that these natural interfaces are also interelement boundaries.

#### 4.3 Modified Forcing Potentials

Through various manipulations and assumptions detailed in Ref. 10 the forcing potential  $P^d$  may be transformed to

$$P^d(\tilde{u}, \tilde{\sigma}, \tilde{d}) = (b, \tilde{u})_V + [\tilde{t}, \tilde{d}]_S + [\tilde{\sigma}_n, \tilde{u} - \tilde{d}]_S. \quad (19)$$

where the all-important surface dislocation integral is taken over  $S$  rather than  $S_i$ . One of the assumptions is that displacement boundary conditions, Eq. 6, are exactly satisfied on  $S_d$ . This expression of  $P^d$  is used in the sequel. A similar technique can be used to adjust  $P^t$ , but that modified formula will not be required in what follows.

#### 4.4 Complete Functionals

Complete elasticity functionals are obtained by combining the generalised strain energy with one of the forcing potentials. For example, the  $d$  and  $t$  generalised versions of the Hu-Washizu functional are

$$\Pi_W^d = U_W - P^d, \quad \Pi_W^t = U_W - P^t. \quad (20)$$

where  $U_W$  is obtained by setting  $j_{22} = j_{13} = 1$ ,  $j_{12} = -1$ , others zero, in Eq. 15.

### 5. MATRIX REPRESENTATION OF ELASTICITY FUNCTIONALS

The generalized strain energy of Eq. 15 can be presented in the matrix form

$$U = \frac{1}{2} \int_V \langle \tilde{\sigma} \quad \sigma^e \quad \sigma^u \rangle \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ & j_{22} & j_{23} \\ \text{symm} & & j_{33} \end{bmatrix} \begin{Bmatrix} e^e \\ \tilde{e} \\ e^u \end{Bmatrix} dV. \quad (21)$$

The symmetric matrix<sup>2</sup>

$$J = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{12} & j_{22} & j_{23} \\ j_{13} & j_{23} & j_{33} \end{bmatrix} \quad (22)$$

characterizes the volume portion of the variational principle. Using the relations  $\sigma^e = Ee$ ,  $\sigma^u = ED\tilde{u}$ ,  $e^e = E^{-1}\sigma$ , and  $e^u = D\tilde{u}$ , the above integral may be rewritten in terms of the independent fields as

$$U = \frac{1}{2} \int_V \langle \tilde{\sigma} \quad \tilde{e} \quad \tilde{u} \rangle \begin{bmatrix} j_{11}E^{-1} & j_{12}I & j_{13}D \\ j_{12}I & j_{22}E & j_{23}ED \\ j_{13}D^T & j_{23}D^TE & j_{33}D^TED \end{bmatrix} \begin{Bmatrix} \tilde{\sigma} \\ \tilde{e} \\ \tilde{u} \end{Bmatrix} dV. \quad (23)$$

<sup>2</sup> To justify the symmetry of  $J$  note, for example, that  $j_{13}(\tilde{\sigma}, e^u)_V = \frac{1}{2}j_{13}(\tilde{\sigma}, e^u)_V + \frac{1}{2}j_{13}(e^e, \sigma^u)_V$ , and so on.

### 5.1 First Variation of Generalized Strain Energy

The first variation of Eq. 15 may be presented as

$$\delta U = (\Delta e, \delta \tilde{e})_V + (\Delta \sigma, \delta \tilde{e})_V - (\text{div } \sigma', \delta \tilde{u})_V + [\sigma'_n, \delta \tilde{u}]_S, \quad (24)$$

where

$$\begin{aligned} \Delta e &= j_{11} e' + j_{12} \tilde{e} + j_{13} e'', \\ \Delta \sigma &= j_{12} \tilde{e} + j_{22} \sigma' + j_{23} \sigma'', \\ \sigma' &= j_{13} \tilde{e} + j_{23} \sigma' + j_{33} \sigma''. \end{aligned} \quad (25)$$

The last two terms combine with contributions from the forcing potential variation. For example, if  $P \equiv P^e$  the complete variation of  $\Pi^e = U - P^e$  is

$$\delta \Pi^e = (\Delta e, \delta \tilde{e})_V + (\Delta \sigma, \delta \tilde{e})_V - (\text{div } \sigma' + b, \delta \tilde{u})_V + [\sigma'_n - \bar{t}, \delta \tilde{u}]_S - [\tilde{u} - \bar{d}, \delta \tilde{e}_n]_{S_d}. \quad (26)$$

Using  $P^d$  or  $P^t$  does not change the volume terms. The first variations of  $\Pi^d$  and  $\Pi^t$  are studied in Refs. 9-11 for a more restrictive class of functionals, namely  $\Pi_7$ . The Euler equations associated with the volume terms

$$\Delta e = 0, \quad \Delta \sigma = 0, \quad \text{div } \sigma' + b = 0, \quad (27)$$

are independent of the forcing potential. A "weighted residual" interpretation of Eqs. 27 in terms of the field equations is given in §5.4. For the moment we note that for consistency of the Euler equations with the field equations of §2 we must have  $\Delta e = 0$ ,  $\Delta \sigma = 0$  and  $\sigma' = \sigma$  if the assumed stress and strain fields reduce to the exact ones. Consequently

$$\begin{aligned} j_{11} + j_{12} + j_{13} &= 0, \\ j_{12} + j_{22} + j_{23} &= 0, \\ j_{13} + j_{23} + j_{33} &= 1. \end{aligned} \quad (28)$$

Because of these constraints, the maximum number of independent parameters that define the entries of  $J$  is three.

### 5.2 Specific Functionals

Expressions of  $J$  for some classical and parametrised variational principles of elasticity are tabulated below. The subscript of  $J$  is used to identify the functionals, which are listed roughly in order of ascending complexity. The fields included in parentheses after the functional name are those subject to independent variations in  $V$ .

Potential energy ( $\tilde{u}$ ):

$$J_P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (29)$$

Stress-displacement Reissner, also called Hellinger-Reissner, ( $\tilde{\sigma}, \tilde{u}$ ):

$$J_R = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (30)$$

Unnamed stress-displacement functional listed on p. 116 of Ref. 18 ( $\tilde{\sigma}, \tilde{u}$ ):

$$J_U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}. \quad (31)$$

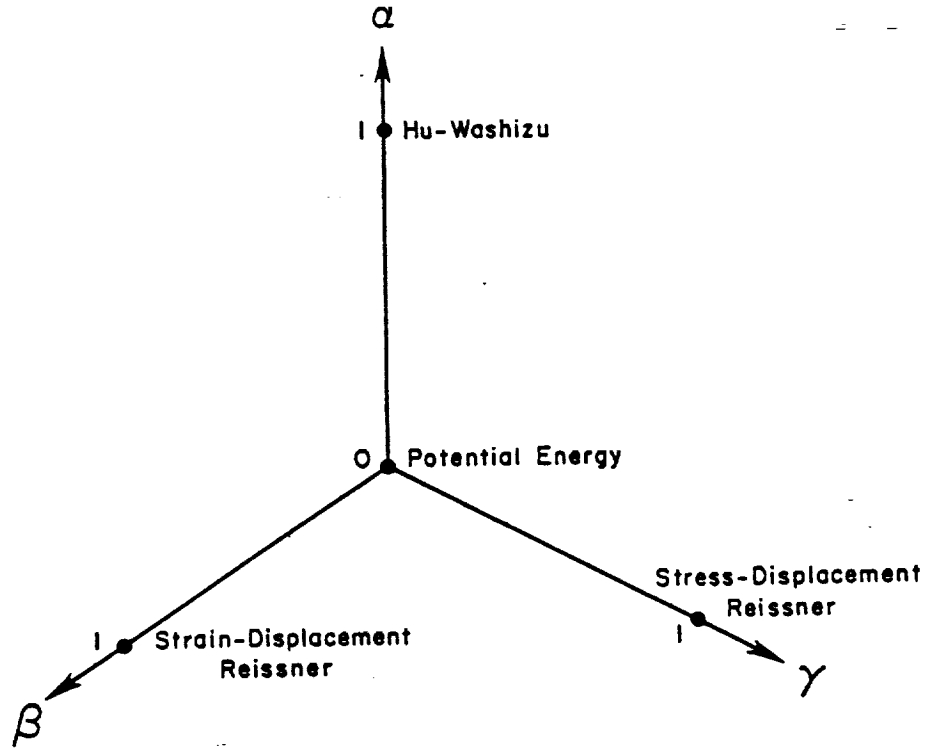


Figure 4 Graphical representation of the  $J_{\alpha\beta\gamma}$  functionals

Strain-displacement Reissner-type as listed on p. 116 of Ref. 18 ( $\tilde{\epsilon}, \tilde{u}$ ):

$$J_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (32)$$

Hu-Washizu<sup>3</sup> ( $\tilde{\sigma}, \tilde{\epsilon}, \tilde{u}$ ):

$$J_W = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (33)$$

One-parameter stress-displacement family ( $\tilde{\sigma}, \tilde{u}$ ) that includes  $U_P$ ,  $U_R$  and  $U_U$  as special cases (Refs. 8-10)

$$J_\gamma = \begin{bmatrix} -\gamma & 0 & \gamma \\ 0 & 0 & 0 \\ \gamma & 0 & 1-\gamma \end{bmatrix}. \quad (34)$$

One-parameter strain-displacement family ( $\tilde{\epsilon}, \tilde{u}$ ) that includes  $U_P$  and  $U_S$  as special cases (Ref. 9)

$$J_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\beta & \beta \\ 0 & \beta & 1-\beta \end{bmatrix}. \quad (35)$$

<sup>3</sup> There are several functionals that carry this name, transformable from one to another through integration by parts. That corresponding to  $J_W$  is the third form listed in §2.3 of Ref. 24.

Two-parameter family  $(\tilde{\sigma}, \tilde{\epsilon}, \tilde{u})$  that includes  $U_\beta$  and  $U_\gamma$  as special cases (Ref. 9):

$$\begin{aligned} J_{\beta\gamma} &= (1-\beta)J_\gamma + (1-\gamma)J_\beta - (1-\beta-\gamma)J_P \\ &= \begin{bmatrix} -\gamma(1-\beta) & 0 & \gamma(1-\beta) \\ 0 & -\beta(1-\gamma) & \beta(1-\gamma) \\ \gamma(1-\beta) & \beta(1-\gamma) & 1-\beta-\gamma+2\beta\gamma \end{bmatrix}. \end{aligned} \quad (36)$$

Three-parameter  $(\alpha, \beta, \gamma)$  family  $(\tilde{\sigma}, \tilde{\epsilon}, \tilde{u})$  that includes  $U_W$  and  $U_{\beta\gamma}$  as special cases (Ref. 9):

$$\begin{aligned} J_{\alpha\beta\gamma} &= \alpha J_W + (1-\alpha)J_{\beta\gamma} \\ &= \begin{bmatrix} -\gamma(1-\beta)(1-\alpha) & -\alpha & \alpha + \gamma(1-\beta)(1-\alpha) \\ -\alpha & \alpha - \beta(1-\gamma)(1-\alpha) & \beta(1-\gamma)(1-\alpha) \\ \alpha + \gamma(1-\beta)(1-\alpha) & \beta(1-\gamma)(1-\alpha) & (1-\beta-\gamma+2\beta\gamma)(1-\alpha) \end{bmatrix}. \end{aligned} \quad (37)$$

The last form, which contains three independent parameters, supplies all matrices  $J$  that satisfy the constraints of Eq. 28. It yields stress-displacement functionals for  $\alpha = \beta = 0$ , strain-displacement functionals for  $\alpha = \gamma = 0$ , and three-field (stress-strain-displacement) functionals otherwise. A graphic representation of this functional in  $(\alpha, \beta, \gamma)$  space is given in Fig. 4.

The specialization  $\gamma = 1, \beta = 0$  of  $J_{\alpha\beta\gamma}$  is of particular interest:

$$J_\alpha = \begin{bmatrix} \alpha-1 & -\alpha & 1 \\ -\alpha & \alpha & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (38)$$

The associated functional  $\Pi_\alpha$  might be called the "generalised" Hu-Washizu functional since it reduces to  $J_W$  for  $\alpha = 1$ . But because of its special relation with the ANDES formulation covered in §8-11,  $\Pi_\alpha$  will be herein referred to as the ANDES functional.

### 5.3 Energy Balancing

A prime motivation for introducing the  $j$  coefficients as free parameters is optimisation of finite element performance in the balanced-stiffness sense of Table 1. The determination of "best" parameters for specific elements relies on the concept of energy balance. Let  $\mathcal{U}(\epsilon) = \frac{1}{2}(\mathbf{E}\epsilon, \epsilon)_V$  denote the strain energy associated with the strain field  $\epsilon$ . If  $\mathbf{E}$  is positive definite,  $\mathcal{U}(\epsilon)$  is nonnegative. We may decompose the generalized strain energy into the following sum of strain energies:

$$U = \mathcal{U}(\mathbf{e}^u) + w_1 \mathcal{U}(\mathbf{e}^\sigma - \tilde{\mathbf{e}}) + w_2 \mathcal{U}(\tilde{\mathbf{e}} - \mathbf{e}^u) + w_3 \mathcal{U}(\mathbf{e}^u - \mathbf{e}^\sigma), \quad (39)$$

where  $\mathcal{U}_P(\mathbf{e}^u) = U_P$  is the usual strain energy, and<sup>4</sup>

$$w_1 = \frac{1}{2}(j_{11} + j_{22} - j_{33} + 1), \quad w_2 = \frac{1}{2}(-j_{11} + j_{22} + j_{33} - 1), \quad w_3 = \frac{1}{2}(j_{11} - j_{22} + j_{33} - 1). \quad (40)$$

Eq. (39) is equivalent to decomposing  $J$  into the sum of four rank-one matrices:

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + w_1 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + w_3 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \quad (41)$$

Decompositions of this nature can be used to derive energy-balanced finite elements by considering element "patches" under simple load systems. This technique is discussed for the one-parameter functionals generated by  $J_\gamma$  in Refs. 6, 8-11. It is important to note that the  $j$  coefficients *may vary from element to element*.

<sup>4</sup> As shown in §5.4, these coefficients may be interpreted as field-equation-residual weights, hence the notation. It is conjectured that for stability the  $j$  coefficients should be confined so that  $w_i \geq 0$ , but this remains to be proven.

#### 5.4 Interpretation of Euler Equations

Eqs. 27 gain physical meaning if they are rewritten as

$$\begin{aligned}\Delta e &= w_1(e^\sigma - \bar{e}) + w_3(e^\sigma - e^\sigma) = 0, \\ \Delta \sigma &= w_1(\bar{\sigma} - \sigma^\sigma) + w_2(\sigma^\sigma - \sigma^\sigma) = 0, \\ \text{div } \sigma' &= \text{div} [\sigma^\sigma + w_2(\sigma^\sigma - \sigma^\sigma) + w_3(\sigma^\sigma - \bar{\sigma})] = -b,\end{aligned}\tag{42}$$

where the  $w_i$  are given by Eqs. 40. But  $e^\sigma - \bar{e} = E^{-1}\bar{\sigma} - \bar{e} = 0$  as well as  $\bar{\sigma} - \sigma^\sigma = \bar{\sigma} - E\bar{e} = 0$  are representations of the constitutive equations, Eqs. 2. Likewise,  $\sigma^\sigma - \sigma^\sigma = E(\bar{e} - D\bar{u}) = 0$  is a representation of the strain-displacement equations, Eqs. 1. Finally,  $e^\sigma - e^\sigma = D\bar{u} - E^{-1}\bar{\sigma} = 0$ , as well as  $\sigma^\sigma - \bar{\sigma} = 0$ , are combinations of Eqs. 1-2. Thus we conclude that the Euler equations  $\Delta e = 0$  and  $\Delta \sigma = 0$  are weighted forms of the kinematic and constitutive field equations. On the other hand,  $\text{div } \sigma' + b = 0$  is a weighted combination of the equilibrium equations, Eqs. 3, and the other two.

If the  $j$  coefficients are such that a weight vanishes (see also footnote 4), that particular field equation drops out from the Euler equations and must be viewed as being satisfied *a priori*. For example, in the potential energy functional,  $w_1 = w_2 = w_3 = 0$  and only the equilibrium condition in terms of  $\sigma^\sigma$  remains in the Euler equations. This interpretation points the way for constructing  $U$  of Eq. 15 by the method of weighted residuals.

#### 5.5 Functionals without Independent Displacements

The foregoing theory applies to functionals where the displacements  $u$  are independently varied. Although this case includes the more practically important functionals for our purposes, for completeness we present here the general parametrization of stress-strain functionals. Decompose  $U$  of Eq. 15 as  $U_c + U_u$ , where  $U_u$  contains the strain energy due to displacement-derived strains:

$$U_u = (j_{13}\bar{\sigma} + j_{23}\sigma^\sigma + \frac{1}{2}j_{33}\sigma^\sigma, e^\sigma)_V = (\text{div } \sigma', u)_V - [\sigma'_n, u]_S.\tag{43}$$

If we now assume that the equilibrium equations  $\text{div } \sigma + b = 0$  and traction boundary conditions  $\sigma_n = \bar{t}$  hold *a priori*,  $U_u$  may be dropped and we are left with the *generalized complementary energy functional*

$$U \rightarrow U_c = \frac{1}{2}j_{11}(\bar{\sigma}, e^\sigma)_V + j_{12}(\bar{\sigma}, \bar{e})_V + \frac{1}{2}j_{22}(\sigma^\sigma, \bar{e})_V.\tag{44}$$

Taking account of the *a priori* conditions, the first variation becomes

$$\delta U_c = (j_{11}e^\sigma + j_{12}\bar{e} + e^\sigma, \delta \bar{\sigma})_V + (j_{12}\sigma^\sigma + j_{22}\sigma^\sigma, \delta \bar{e})_V,\tag{45}$$

and for consistency we must have  $j_{11} + j_{12} = -1$ ,  $j_{12} + j_{22} = 0$ . It follows that  $U_c$  may be represented as in the matrix form of Eq. 21 with a  $J$  that depends on a single parameter:

$$J_\rho = \begin{bmatrix} \rho - 1 & -\rho & 0 \\ -\rho & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}.\tag{46}$$

Here  $\rho = 0$  gives the classical principle of total complementary energy whereas  $\rho = 1$  gives the functional  $N(\bar{\sigma}, \bar{e})$  listed on p. 117 of Ref. 18.

### 6. FINITE ELEMENT DISCRETIZATION

In this section assumptions invoked in the finite element discretization of the functional  $\Pi^d$  for arbitrary  $J$  are stated. Following usual practice in finite element work, the components of stresses and strains are arranged as one-dimensional arrays while the elastic moduli in  $E$  are arranged as a square symmetric matrix. In the sequel, and unless otherwise noted, we consider an *individual element* of volume  $V$  and surface  $S : S_t \cup S_d \cup S_i$ , where  $S_i$  is the portion of the boundary in common with other elements.

### 6.1 Boundary Displacement Assumption

The boundary displacement assumption is

$$[\bar{d} = N_d v]_S. \quad (47)$$

Here matrix  $N_d$  collects boundary shape functions for the boundary displacements  $d$  while vector  $v$  collects the "visible" degrees of freedom of the element, also called the *connectors*. These displacements must be unique on common element boundaries. This continuity condition is met if the displacement of a common boundary portion is uniquely specified by degrees of freedom located on that boundary. There are no derived fields associated with  $\bar{d}$ .

### 6.2 Internal Displacement Assumption

The displacement assumption in the interior of the element is

$$(\bar{u} = N_u q)_V, \quad (48)$$

where matrix  $N_u$  collects the internal displacement shape functions and vector  $q$  collects generalised coordinates for the internal displacements. The assumed  $\bar{u}$  need not be continuous across interelement boundaries. The displacement derived fields are

$$(e^* = DNq = Bq)_V, \quad (\sigma^* = EBq)_V. \quad (49)$$

To link up with the FF and ANS formulations, we proceed to break up the internal displacement field as follows. The assumed  $\bar{u}$  is decomposed into rigid body, constant strain, and higher order displacements:

$$\bar{u} = N_r q_r + N_c q_c + N_h q_h. \quad (50)$$

Applying the strain operator  $D = \frac{1}{2}(\nabla + \nabla^T)$  to  $\bar{u}$  we get the associated strain field:

$$e^* = DN_r q_r + DN_c q_c + DN_h q_h = B_r q_r + B_c q_c + B_h q_h. \quad (51)$$

But  $B_r = DN_r$  vanishes because  $N_r$  contains only rigid-body modes. We are also free to select  $B_c = DN_c$  to be the identity matrix  $I$  if the generalised coordinates  $q_c$  are identified with the mean (volume-averaged) strain values  $\bar{e}^*$ . Consequently Eq. 51 simplifies to

$$e^* = \bar{e}^* + e_h^* = \bar{e}^* + B_h q_h, \quad (52)$$

in which

$$q_c \equiv \bar{e}^* = (e^*)_V / v, \quad (B_h)_V = 0, \quad (53)$$

where  $v = (1)_V$  is the element volume measure. The second relation is obtained by integrating both sides of Eq. 52 over  $V$  and noting that  $q_h$  is arbitrary. It says that the mean value of the higher-order displacement-derived strains (also called the *deviatoric* displacement-derived strains) is zero over the element.

### 6.3 Stress Assumption

The stress field will be assumed to be constant over the element:

$$(\bar{\sigma} = \sigma)_V. \quad (54)$$

This assumption is sufficient to construct high-performance elements based on the free formulation (FF) developed in Refs. 4-8. As discussed in Ref. 11, the inclusion of higher order stress modes (deviatoric stresses) in Eq. 56 is computationally effective if these modes are divergence free, but such a requirement makes extension to geometrically nonlinear problems difficult. The only derived field is

$$(e^\sigma = \bar{e}^\sigma = E^{-1} \bar{\sigma})_V. \quad (55)$$

#### 6.4 Strain Assumptions

The assumed strain field  $\tilde{\epsilon}$  is split into a mean constant strain  $\bar{\epsilon}$  and a higher order variation (the deviatoric strains):

$$(\tilde{\epsilon} = \bar{\epsilon} + e_d = \bar{\epsilon} + A_d a)_V, \quad (56)$$

where  $\bar{\epsilon} = (\tilde{\epsilon})_V / v$ , matrix  $A_d$  collects deviatoric strain modes with mean zero value over the element:

$$(A_d)_V = 0, \quad (57)$$

and  $a$  collects the corresponding strain mode amplitudes. The only derived field is

$$(\sigma^e = E\tilde{\epsilon} = E\bar{\epsilon} + EA_d a)_V. \quad (58)$$

### 7. UNCONSTRAINED FINITE ELEMENT EQUATIONS

For simplicity we shall assume that all elastic moduli in  $E$  are constant over the element. Inserting the above assumptions into  $\Pi^d$  with the modified forcing potential of Eq. 19 we obtain a quadratic algebraic form, which is block-sparse on account of the conditions stated in Eqs. 53 and 57. Rendering this form stationary yields the finite element equations

$$\begin{bmatrix} j_{11}vE^{-1} & j_{12}vI & 0 & -P_r^T & j_{13}vI - P_u^T & -P_h^T & L^T \\ j_{12}vI & j_{22}vE & 0 & 0 & j_{23}vI & 0 & 0 \\ 0 & 0 & j_{22}K_{dd} & 0 & 0 & j_{23}R^T & 0 \\ -P_r & 0 & 0 & 0 & 0 & 0 & 0 \\ j_{13}vI - P_u & j_{23}vI & 0 & 0 & j_{33}vE & 0 & 0 \\ -P_h & 0 & j_{13}R & 0 & 0 & j_{33}K_{qh} & 0 \\ L & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \bar{\epsilon} \\ a \\ q_r \\ \bar{e}^u \\ q_h \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{qr} \\ f_{qu} \\ f_{qh} \\ f_v \end{bmatrix}, \quad (59)$$

where

$$\begin{aligned} K_{qh} &= (B_h^T E B_h)_V = K_{qh}^T, \quad K_{dd} = (A_d^T E A_d)_V = K_{dd}^T, \quad R = (B_h^T E A_d)_V, \\ L &= [N_{dn}^T]_S, \quad P_r = [N_{rn}^T]_S, \quad P_u = [N_{un}^T]_S, \quad P_h = [N_{hn}^T]_S, \\ f_r &= (N_r^T b)_V, \quad f_q = (N_q^T b)_V, \quad f_h = (N_h^T b)_V, \quad f_v = [N_d^T \hat{t}]_S, \end{aligned} \quad (60)$$

in which  $N_{dn}$  denotes the projection of shape functions  $N_d$  on the exterior normal  $n$ , and similarly for  $N_r$ ,  $N_u$  and  $N_h$ . Those coefficient matrix entries that do not depend on the  $j$  coefficients come from the last boundary term in Eq. 19.

#### 7.1 The P Matrices

Application of the divergence theorem to the work of the mean stress on  $e^u$  yields

$$\begin{aligned} (\bar{\sigma}, e^u)_V &= (\bar{\sigma}, \bar{e}^u + B_h q_h)_V = v \bar{\sigma}^T \bar{e}^u + \bar{\sigma}^T (B_h)_V q_h = v \bar{\sigma}^T \bar{e}^u \\ &= [\bar{\sigma}_n, \bar{u}]_S = [\bar{\sigma}_n, N_r q_r + N_u \bar{e}^u + N_h q_h]_S = \bar{\sigma}^T (P_r q_r + P_u \bar{e}^u + P_h q_h). \end{aligned} \quad (61)$$

Hence  $P_r = 0$ ,  $P_u = vI$ ,  $P_h = 0$ , and the element equations simplify to

$$\begin{bmatrix} j_{11}vE^{-1} & j_{12}vI & 0 & 0 & (j_{13} - 1)vI & 0 & L^T \\ j_{12}vI & j_{22}vE & 0 & 0 & j_{23}vI & 0 & 0 \\ 0 & 0 & j_{22}K_{dd} & 0 & 0 & j_{23}R^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (j_{13} - 1)vI & j_{23}vI & 0 & 0 & j_{33}vE & 0 & 0 \\ 0 & 0 & j_{13}R & 0 & 0 & j_{33}K_{qh} & 0 \\ L & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \bar{\epsilon} \\ a \\ q_r \\ \bar{e}^u \\ q_h \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{qr} \\ f_{qu} \\ f_{qh} \\ f_v \end{bmatrix}. \quad (62)$$

The simplicity of the  $P$  matrices is essentially due to the mean-plus-deviator splitting of Eq. 52 for  $e^u$ . If this decomposition is not enforced,  $P_r = 0$  but  $P_e = (B_e)v = vB_e$  and  $P_h = (B_h)v$ .

## 8. KINEMATIC CONSTRAINTS

The "tricks" we shall consider here are *kinematic constraints* that play a key role in the development of high-performance FF and ANS elements. These are matrix relations between kinematic quantities that are established *independently* of the variational equations. Two types of relations will be studied.

### 8.1 Constraints Between Internal and Boundary Displacements

Relations linking the generalized coordinates  $q$  of Eq. 48 and the connectors  $v$  were introduced by Bergan and coworkers in conjunction with the free formulation (FF) of finite elements (see Ref. 5). For simplicity we shall assume that the number of freedoms in  $v$  and  $q$  is the same; removal of this restriction is studied in Ref. 11. By collocation of  $u$  at the element node points one easily establishes the relation

$$v = G_r q_r + G_e q_e + G_h q_h = Gq, \quad (63)$$

where  $G$  is a square transformation matrix that will be assumed to be nonsingular. On inverting this relation we obtain

$$q = G^{-1} = Hv, \quad \text{or} \quad q = \begin{Bmatrix} q_r \\ \bar{e}^u \\ q_h \end{Bmatrix} = \begin{Bmatrix} H_r \\ H_e \\ H_h \end{Bmatrix} v. \quad (64)$$

The following relations between  $L$  (defined in Eq. 60) and the above submatrices hold as a consequence of the individual element test described in §9.3:

$$L^T G_r = 0, \quad L^T G_e = vL, \quad vH_e = L^T. \quad (65)$$

If the splitting of Eq. 52 is not enforced, however, the last two become

$$L^T G_e = vB_e, \quad P_e H_e + P_h H_h = L^T \quad \text{or} \quad P_e = L^T G_e. \quad (66)$$

Since  $P_e = vB_e$ , these relations coalesce (see Ref. 5).

### 8.2 Constraints Between Assumed Deviatoric Strains and Boundary Displacements

Constraints linking  $\bar{e}$  to  $v$  are of fundamental importance in the assumed natural strain (ANS) formulation. The effect of these constraints in a variational framework is analysed in Refs. 15 and 16. In the present study we depart from previous work in that *only the deviatoric strains,  $e_d$ , are assumed linked to  $v$  whereas the mean strains  $\bar{e}$  are obtained variationally*. Consequently we shall postulate the following relation between assumed deviatoric strain amplitudes and nodal displacement connectors:

$$a = Qv, \quad (67)$$

where  $Q$  is generally a rectangular matrix determined by collocation, least squares or other fitting methods. An example showing the construction of  $Q$  is given in §11.4. The individual element test described in §9.3 requires that  $Q$  be orthogonal to  $G_r$  and  $G_e$ :

$$QG_r = 0, \quad QG_e = 0. \quad (68)$$

### 8.3 Limitation Principles

Strain assumptions made concurrently with displacement assumptions are confined by *limitation principles* similar to those stated by Fraeijs de Veubeke for stress-displacement mixed elements (Ref. 13). This issue was discussed in Ref. 15 for a more restricted strain-displacement hybrid formulation. Limitation principles for the general formulation presented here remain to be studied.

## 9. VISIBLE STIFFNESS EQUATIONS

On enforcing the constraints  $\mathbf{a} = \mathbf{Q}\mathbf{v}$ ,  $\mathbf{q}_r = \mathbf{H}_r\mathbf{v}$ ,  $\mathbf{q}_e = \mathbf{H}_e\mathbf{v} = \mathbf{v}^{-1}\mathbf{L}^T\mathbf{v}$ , and  $\mathbf{q}_h = \mathbf{H}_h\mathbf{v}$ , through Lagrange multiplier vectors  $\lambda_a$ ,  $\lambda_r$ ,  $\lambda_e$ , and  $\lambda_h$ , respectively, we get the augmented finite element equations

$$\begin{bmatrix} j_{11}\mathbf{vE}^{-1} & j_{12}\mathbf{vI} & 0 & 0 & (j_{13}-1)\mathbf{vI} & 0 & 0 & 0 & 0 & 0 & \mathbf{L}^T \\ j_{12}\mathbf{vI} & j_{22}\mathbf{vE} & 0 & 0 & j_{23}\mathbf{vI} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j_{22}\mathbf{K}_{ed} & 0 & 0 & j_{23}\mathbf{R}^T & -\mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 \\ (j_{13}-1)\mathbf{vI} & j_{23}\mathbf{vI} & 0 & 0 & j_{23}\mathbf{vE} & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 \\ 0 & 0 & j_{23}\mathbf{R} & 0 & 0 & j_{23}\mathbf{K}_{eh} & 0 & 0 & 0 & -\mathbf{I} & 0 \\ 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q} \\ 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{H}_r \\ 0 & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & \mathbf{v}^{-1}\mathbf{L}^T \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & \mathbf{H}_h \\ \mathbf{L} & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}^T \mathbf{H}_r^T \mathbf{v}^{-1} \mathbf{L} \mathbf{H}_h^T & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{v}} \\ \bar{\mathbf{a}} \\ \mathbf{q}_r \\ \mathbf{q}_e \\ \mathbf{q}_h \\ \lambda_a \\ \lambda_r \\ \lambda_e \\ \lambda_h \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \mathbf{f}_{qr} \\ \mathbf{f}_{qe} \\ \mathbf{f}_{qh} \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathbf{f}_e \end{Bmatrix} \quad (69)$$

Condensation of all degrees of freedom except  $\mathbf{v}$  yields the visible<sup>5</sup> element stiffness equations

$$\mathbf{K}\mathbf{v} = (\mathbf{K}_b + \mathbf{K}_h)\mathbf{v} = \mathbf{f}, \quad (70)$$

where

$$\mathbf{K}_b = \mathbf{v}^{-1} \mathbf{L} \mathbf{E} \mathbf{L}^T, \quad (71)$$

$$\mathbf{K}_h = j_{23} \mathbf{H}_h^T \mathbf{K}_{eh} \mathbf{H}_h + j_{23} (\mathbf{H}_h^T \mathbf{R} \mathbf{Q} + \mathbf{Q}^T \mathbf{R}^T \mathbf{H}_h) + j_{22} \mathbf{Q}^T \mathbf{K}_{ed} \mathbf{Q}, \quad (72)$$

$$\mathbf{f} = \mathbf{f}_e + \mathbf{H}_r^T \mathbf{f}_{qr} + \mathbf{v}^{-1} \mathbf{L}^T \mathbf{f}_{qe} + \mathbf{H}_h^T \mathbf{f}_{qh}. \quad (73)$$

Following the nomenclature of the free formulation, we shall call  $\mathbf{K}_b$  the *basic stiffness matrix* and  $\mathbf{K}_h$  the *higher order stiffness matrix*.

### 9.1 Relation to Previous HP Element Formulations

If  $\mathbf{J} = \mathbf{J}_\gamma$  of Eq. 33,  $j_{33} = 1 - \gamma$ ,  $j_{22} = j_{23} = 0$ , and we recover the scaled *free formulation* stiffness equations considered in Refs. 6, 8 and 10:

$$\mathbf{K}_h = (1 - \gamma) \mathbf{H}_h^T \mathbf{K}_{eh} \mathbf{H}_h, \quad 1 - \gamma > 0. \quad (74)$$

On the other hand, if we take  $\mathbf{J} = \mathbf{J}_\alpha$  as given in Eq. 38,  $j_{22} = \alpha$ ,  $j_{33} = j_{23} = 0$  and we obtain

$$\mathbf{K}_h = \alpha \mathbf{Q}^T \mathbf{K}_{ed} \mathbf{Q}, \quad \alpha > 0, \quad (75)$$

which is similar to the stiffness produced by the ANS hybrid variational formulation studied in Refs. 15-16, in which the forcing potential  $P^d$  was used instead of  $P^d$ . The variant of ANS

<sup>5</sup> The qualifier *visible* emphasizes that these are the stiffness equations other elements "see", and consequently are the only ones that matter insofar as computer implementation on a displacement-based finite element program.

considered herein will be called the *assumed natural deviatoric strain* (ANDES) formulation in the sequel. The name is apt in the sense that what is being assumed are deviatoric rather than total strains, and that this assumption only affects the higher order stiffness.

But the term with coefficient  $j_{23}$  in Eq. 72 is new. It may be viewed as coupling the FF and ANDES formulations. It is not known at this time whether Eqs. 70-73 represent the most general structure of the visible stiffness equations of HP elements.

## 9.2 Recovery of Element Fields

For simplicity suppose that the *body forces vanish* and so do  $f_{qr}$ ,  $f_{qo}$  and  $f_{qh}$  on account of Eqs. 60. If  $v$  is known following a finite element solution of the assembled system, solving Eqs. 69 for the internal degrees of freedom yields

$$\begin{aligned} \bar{\epsilon} &= v^{-1} L^T v, \quad \bar{\sigma} = E \bar{\epsilon}, \quad a = Qv, \quad q_r = H_r v, \quad \bar{e}^* = \bar{\epsilon}, \quad q_h = H_h v, \\ \lambda_o &= (j_{22} K_{od} Q + j_{33} R^T H_h) v, \quad \lambda_r = 0, \quad \lambda_o = 0, \quad \lambda_h = (j_{23} R Q + j_{33} K_{qh} H_h) v. \end{aligned} \quad (76)$$

It is seen that the mean strains  $\bar{\epsilon}$ ,  $\bar{e}^*$  and  $\bar{e}^* = E^{-1} \bar{\sigma}$  coincide, and of course so do the mean stresses. But if the body forces do not vanish the mean stresses and mean strains recovered from different fields will not generally agree.

It is also worthwhile to note that a nonzero Lagrange multiplier vector flags a deviation of the associated fields from the *variationally consistent* fields that would result on using the unconstrained Eqs. 62 without "tricks".

## 9.3 The Individual Element Test

To conclude the general formulation we investigate the conditions under which HP elements based on the foregoing setting pass the individual element test of Bergan and Hanssen described in Refs. 3-6. To carry out the test, assume that the "free floating" element<sup>6</sup> under zero body forces is in a *constant stress state*  $\sigma_0$ , which of course is also the mean stress. Insert the following data in the left-hand side vector of Eq. 69:

$$\begin{aligned} \bar{\sigma} &= \sigma_0 = \bar{\sigma}^*, \quad \bar{\epsilon} = E^{-1} \sigma_0, \quad a_h = 0, \quad q_r = \text{arbitrary}, \quad e^* = \bar{e}^* = E^{-1} \sigma_0, \quad q_h = 0, \\ \lambda_o &= 0, \quad \lambda_r = 0, \quad \lambda_o = 0, \quad \lambda_h = 0, \quad v = G_r q_r + G_o \bar{e}^* = G_r q_r + G_o E^{-1} \sigma_0. \end{aligned} \quad (77)$$

Premultiply by the coefficient matrix, and demand that all terms on the right-hand side vanish but for  $f_o = L \sigma_0$ . Then the orthogonality conditions listed in Eqs. 65 and 68 emerge. This form of the patch test is very strong, and it may well be that relaxing circumstances can be found for specific problems such as shells.

## 10. DISCUSSION

At this point it is useful to recapitulate key points of the previous development, and to connect this material with some of the techniques of Table 2. The chief property of HP elements constructed with present methods is the decomposition of the element stiffness equations displayed in Eq. 70; a property that of course subsists at the assembly level.

The basic stiffness matrix has a *universal* character: as no coefficients  $j$  appear in Eq. 71, clearly  $K_b$  is independent of specific variational principles. Given the constant stress state introduced in Eq. 54,  $K_b$  depends only on the assumed boundary motions. It can be constructed (and programmed) once and for all for each element type. As emphasized in Ref. 5, the main function of  $K_b$  is to provide *convergence*.

The higher order stiffness given in Eq. 72 serves two other functions: *stability* and *accuracy*. The basic stiffness is generally rank-deficient<sup>7</sup> because its rank cannot exceed

<sup>6</sup> Mathematically, the entire element boundary is traction-specified, i.e.,  $S \equiv S_t$ .

<sup>7</sup> Except in simplex elements, for which  $K \equiv K_1$ .

that of  $E$ ; thus a key function of  $K_h$  is to stabilize  $K$  by raising its rank to the correct one. The second function, which has gained importance in recent work, is to increase solution accuracy for coarse grids. Here is where the  $j$  coefficients play the important role noted in §5.3. *These coefficients may vary from element to element*, despite the fact that this variation implies that the variational principle changes from one element to another. Thus the "element mixability" requirement of Table 1 is fulfilled without tears.

### 10.1 The Free Formulation

The present methodology was initially pursued to justify variationally the original FF ( $\gamma = 0$ ) of Ref. 5 as well as the scaled FF ( $\gamma \neq 0$ ) of Refs. 6-8. Thus it is not surprising that those element construction techniques fit naturally in the present variational framework by simply taking  $J = J_\gamma$ . The extended FF described in Ref. 11 aims to remove the restriction that the dimension of vectors  $q$  and  $v$  be the same. One of the techniques advocated to allow  $\dim(q) > \dim(v)$  involves extending Eq. 54 with deviatoric stress assumptions, and thus requires a generalization of Eqs. 59 and 62. Whether such a generalization is practically worthwhile is unclear at this time.

### 10.2 The ANS Formulation

The conventional ANS formulation as presented in Refs. 1 and 19 constructs *total* strain fields  $\bar{\epsilon}$  (not necessarily integrable into displacements  $u^*$ ) gaged through generalized strain coordinates  $a$  as  $\epsilon = Aa$ . These coordinates are eventually linked to the connectors  $v$  via matrix expressions of the form  $a = Qv$ , leading to an element stiffness of the form  $K = Q^T K_a Q$ , where  $K_a$  is the generalized stiffness in terms of  $a$ . The restriction to deviatoric strains in §6.4 is motivated by two interrelated factors: (a) the strain-assumed stiffness "flows" to the higher order stiffness, where it can be naturally scaled by using  $J = J_a$ , and even intermixed with FF contributions as Eq. 72 shows; and (b) the basic stiffness of the element, derived separately, can be used to insure convergence.

### 10.3 Projectors and S/R Integration

The so-called "B-bar" approach is based on expressing the element strains as<sup>\*</sup>

$$\epsilon = \bar{B}v \quad (78)$$

where  $\bar{B}$ , which cuts off the "harmful" portion of  $B^*$ , is constructed by various ad-hoc devices such as strain projection, selective and/or uniform reduced integration. These time-honored schemes are well covered in Ref. 14. They are easily included in the present setting if  $\bar{B}$  admits the decomposition

$$\bar{B} = \bar{B} + A_d Q, \quad (79)$$

where  $Q$  is not position dependent and  $\bar{\epsilon} = \bar{B}v$  provides the mean strains, which are discarded in favor of Eq. 76. This decomposition can be usually carried out in several ways.

## 11. EXAMPLE: A 9-DOF ANDES PLATE BENDING TRIANGLE

The first element constructed with the ANDES formulation is a three-node Kirchhoff plate-bending flat triangle with the usual nine degrees of freedom. The derivation is briefly covered here as it illustrates the essential steps in forming the higher order stiffness of such elements. These steps are outlined in "recipe" form in Table 3. This Table restates the arguments of §6.4 in a more physically oriented sense, which is closely aligned with the terminology of Ref. 19.

<sup>\*</sup> This is a slight variation from the usual notation, necessitated by the use of the single overbar to denote average or mean values.

Table 3 Construction of  $K_h$  by the ANDES Formulation

*Step 1.* Select reference lines (in 2D elements) or reference planes (in 3D elements) where "natural strainage" locations are to be chosen. By appropriate interpolation express the element natural strains  $\tilde{\epsilon}$  in terms of the "strainage readings"  $g$  at those locations:

$$\tilde{\epsilon} = A_n g, \quad (80)$$

where  $\tilde{\epsilon}$  is a strain field in natural coordinates that must include all constant strain states. (For bending elements the term "strains" is to be interpreted in a generalised sense, viz. curvatures.)

*Step 2.* Relate the Cartesian strains  $\bar{\epsilon}$  to the natural strains:

$$\bar{\epsilon} = T\tilde{\epsilon} = TA_n g = Ag \quad (81)$$

at each point in the element. (If  $e \equiv \epsilon$ , or if it is possible to work throughout in natural coordinates, this step is skipped.)

*Step 3.* Split the Cartesian strain field into mean (volume-averaged) and deviatoric strains:

$$\bar{\epsilon} = \bar{\epsilon} + e_d = (\bar{A} + A_d)g, \quad (82)$$

where  $\bar{A} = (TA_n)_V / v$ , and  $e_d = A_d g$  has mean zero value over  $V$ . (This step may also be carried out on the natural strains if  $T$  is constant, as is the case for the element derived here.)

*Step 4.* Relate the natural strainage readings  $g$  to the visible degrees of freedom

$$g = Qv \quad (83)$$

where  $Q$  is a strainage-to-node displacement transformation matrix. Techniques by which this is accomplished vary from element to element and is difficult to state rules that apply to every situation. In the element derived here  $Q$  is constructed by direct interpolation over the reference lines. (In general there is no internal displacement field  $u^e$  such that  $\bar{\epsilon} = Du^e$ , so this step cannot be done by simply integrating the field of Eq. 81 over the element and collocating  $u^e$  at the nodes.)

*Step 5.* The higher-order stiffness matrix is given by

$$K_h = \alpha Q^T K_{ad} Q, \quad \text{where} \quad K_{ad} = \int_V A_d^T E A_d dV, \quad (84)$$

where  $\alpha > 0$  is the scaling coefficient supplied by the functional of Eq. 38.

### 11.1 Geometric Relations

The triangle has *straight sides*. Its geometry is completely defined by the location of its three corners, which are labeled 1,2,3, moving counterclockwise. The triangle is referred to a local Cartesian system  $(x, y)$  which is taken with origin at the centroid  $O$ , whence the corner coordinates  $x_i, y_i$  satisfy the relations  $x_1 + x_2 + x_3 = 0$  and  $y_1 + y_2 + y_3 = 0$ . Coordinate differences are abbreviated by writing  $x_{ij} = x_i - x_j$ , etc. The signed triangle area  $A$  is given by the formulas

$$2A = x_{21}y_{31} - x_{31}y_{21} = x_{32}y_{12} - x_{12}y_{32} = x_{13}y_{23} - x_{23}y_{13}, \quad (85)$$

and we require that  $A > 0$ . We shall also make use of dimensionless triangular coordinates  $\zeta_1, \zeta_2, \zeta_3$  linked by the constraint  $\zeta_1 + \zeta_2 + \zeta_3 = 1$ . The following well known relation between the area and centroid-originated Cartesian coordinates of a straight-sided triangle is noted:

$$\zeta_i = \frac{1}{2A}(x_i y_k - x_k y_j + x y_{jk} + y x_{kj}), \quad (86)$$

where  $i, j$  and  $k$  denote positive cyclic permutations of 1, 2 and 3; for example,  $i = 2, j = 3, k = 1$ . Therefore  $\partial \zeta_i / \partial x = y_{jk} / 2A$  and  $\partial \zeta_i / \partial y = x_{kj} / 2A$ . Other intrinsic dimensions and ratios used below are

$$\ell_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2}, \quad a_{ij} = 2A / \ell_{ij}, \quad b_{ij} = (x_{ij} x_{ik} + y_{ij} y_{ik}) / \ell_{ij}, \quad b_{ji} = \ell_{ij} - b_{ij}, \quad (87)$$

$$\lambda_{ij} = b_{ij} / \ell_{ij} = (x_{ij} x_{ik} + y_{ij} y_{ik}) / (x_{ij}^2 + y_{ij}^2), \quad \lambda_{ji} = b_{ji} / \ell_{ij} = 1 - \lambda_{ij},$$

where  $\ell_{ij}$  denote the triangle side lengths,  $a_{ij}$  are triangle heights,  $b_{ij}$  and  $b_{ji}$  are projections of sides  $ik$  and  $jk$  onto side  $ij$ , respectively, and the  $\lambda$ 's are ratios of these projections to the side lengths.

### 11.2 Displacements, Rotations, Side Coordinates

Since we are dealing with a Kirchhoff element, its displacement field is completely defined by the transverse displacement  $w(x, y) \equiv w(\zeta_1, \zeta_2, \zeta_3)$ , positive upwards. The mid-plane rotations about  $x$  and  $y$  are  $\theta_x = \partial w / \partial y$  and  $\theta_y = -\partial w / \partial x$ . The visible degrees of freedom of the element collected in  $\mathbf{v}$  are

$$\mathbf{v}^T = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3}]. \quad (88)$$

Over the three sides 1-2, 2-3 and 3-1, traversed counterclockwise, we define the dimensionless side coordinates  $\mu_{12}, \mu_{23}$  and  $\mu_{31}$  as follows. Over side 1-2,  $\mu_{12}$  varies from  $\mu_{12} = 0$  at corner 1 to  $\mu_{12} = 1$  at corner 2; thus  $\mu_{12} \equiv \zeta_2$  when  $\zeta_3 = 0$ . Relations for the other sides follow from cyclic permutation of subscripts. Then

$$\begin{aligned} \frac{\partial x}{\partial \mu_{12}} &= x_{21}, & \frac{\partial x}{\partial \mu_{23}} &= x_{32}, & \frac{\partial x}{\partial \mu_{31}} &= x_{13}, \\ \frac{\partial y}{\partial \mu_{12}} &= y_{21}, & \frac{\partial y}{\partial \mu_{23}} &= y_{32}, & \frac{\partial y}{\partial \mu_{31}} &= y_{13}. \end{aligned} \quad (89)$$

### 11.3 Natural Curvatures

The second derivatives of  $w$  with respect to the dimensionless side directions will be called the *natural curvatures* and denoted by  $\chi_{ij} = \partial^2 w / \partial \mu_{ij}^2$ . Note that they have dimensions of displacement. The natural curvatures can be related to the Cartesian plate curvatures  $\kappa_{xx} = \partial^2 w / \partial x^2$ ,  $\kappa_{yy} = \partial^2 w / \partial y^2$  and  $\kappa_{xy} = 2\partial^2 w / \partial x \partial y$ , by chain-rule application of Eq. 89:

$$\chi = \begin{Bmatrix} \chi_{12} \\ \chi_{23} \\ \chi_{31} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 w}{\partial \mu_{12}^2} \\ \frac{\partial^2 w}{\partial \mu_{23}^2} \\ \frac{\partial^2 w}{\partial \mu_{31}^2} \end{Bmatrix} = \begin{bmatrix} x_{21}^2 & y_{21}^2 & x_{21} y_{21} \\ x_{32}^2 & y_{32}^2 & x_{32} y_{32} \\ x_{13}^2 & y_{13}^2 & x_{13} y_{13} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \mathbf{T}^{-1} \kappa. \quad (90)$$

The inverse of this relation is

$$\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23} y_{13} & y_{31} y_{21} & y_{12} y_{32} \\ x_{23} x_{13} & x_{31} x_{21} & x_{12} x_{32} \\ y_{23} x_{31} + x_{32} y_{13} & y_{31} x_{12} + x_{13} y_{21} & y_{12} x_{23} + x_{21} y_{32} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial \mu_{12}^2} \\ \frac{\partial^2 w}{\partial \mu_{23}^2} \\ \frac{\partial^2 w}{\partial \mu_{31}^2} \end{Bmatrix}, \quad (91)$$

or, in matrix form

$$\kappa = T\chi. \quad (92)$$

#### 11.4 Curvature Sampling

The *reference lines* referred to in Table 3 are the three triangle sides. The natural curvatures are assumed to vary linearly over each reference line, an assumption which is obviously consistent with cubic beam-like variations of  $w$  over the sides. A linear variation on each side is determined by two straining sample points, which we chose to be at the corners.

On each triangle side choose the isoparametric coordinates  $\xi_{ij}$  that vary from  $-1$  at corner  $i$  to  $+1$  at corner  $j$ . These are related to the  $\mu_{ij}$  coordinates as  $\xi_{ij} = 2\mu_{ij} - 1$ . Then the natural curvature over side  $ij$  is given by the beam formula

$$\chi_{ij} = \frac{\partial^2 w}{\partial \mu_{ij}^2} = \ell_{ij} \left[ \frac{6\xi_{ij}}{\ell_{ij}} \quad 3\xi_{ij} - 1 \quad \frac{-6\xi_{ij}}{\ell_{ij}} \quad 3\xi_{ij} + 1 \right] \begin{Bmatrix} w_i \\ \theta_{ni} \\ w_j \\ \theta_{nj} \end{Bmatrix}, \quad (93)$$

where  $\theta_n$  denote the rotation about the external normal direction  $n$  on side  $ij$ . Evaluating these relations at the nodes by setting  $\xi_{ij} = \pm 1$  and converting normal rotations to  $x$ - $y$  rotations, we build the transformation

$$\begin{Bmatrix} \chi_{12}|_1 \\ \chi_{12}|_2 \\ \chi_{23}|_2 \\ \chi_{23}|_3 \\ \chi_{31}|_3 \\ \chi_{31}|_1 \end{Bmatrix} = \begin{bmatrix} -6 & -4y_{21} & 4x_{21} & 6 & 2y_{21} & -2x_{21} & 0 & 0 & 0 \\ 6 & 2y_{21} & -2x_{21} & -6 & 4y_{21} & -4x_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & -4y_{32} & 4x_{32} & 6 & -2y_{32} & 2x_{32} \\ 0 & 0 & 0 & 6 & 2y_{32} & -2x_{32} & -6 & 4y_{32} & -4x_{32} \\ 6 & -2y_{13} & 2x_{13} & 0 & 0 & 0 & -6 & -4y_{13} & 4x_{13} \\ -6 & 4y_{13} & -4x_{13} & 0 & 0 & 0 & -6 & 2y_{13} & -2x_{13} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \end{Bmatrix} \quad (94)$$

The left hand side is the natural straining reading vector called  $g$  in Table 3 and so we can express this as the matrix relation

$$g = Qv. \quad (95)$$

#### 11.5 Curvature Interpolation

The six gage readings collected in  $g$  provide curvatures along the 3 triangle sides directions at two corners. But 9 values are needed to recover the complete curvature field over the element. The 3 additional values are the natural curvatures at the missing corner. We obtain these values by adopting the following rule: *Cylindrical bending with linearly varying curvature along a side direction is to be exactly represented.* Another way of stating this is that the side curvature  $\chi_{ij}$  is to be constant along lines normal to side  $ij$ . This makes the element insensitive to bad aspect ratios on "strip bending" if each element has a side oriented in the direction of the strip.

To apply this rule consider side 1-2. The natural curvature  $\chi_{12} = \partial^2 w / \partial \mu_{12}^2$  along this side is defined at nodes 1 and 2 by the first two rows of Eq. 94. For node 3 take

$$\chi_{12}|_3 = \frac{\partial^2 w}{\partial \mu_{12}^2} \Big|_3 = \lambda_{12} \chi_{12}|_1 + \lambda_{21} \chi_{12}|_2, \quad (96)$$

where  $\lambda_{12}$  and  $\lambda_{21}$  are defined in Eq. 87. Since we now know the values of  $\chi_{12} = \partial^2 w / \partial \mu_{12}^2$  at the three corners we can use the standard linear interpolation over the entire triangle:

$$\chi_{12} = \chi_{12}|_1 \xi_1 + \chi_{12}|_2 \xi_2 + \chi_{12}|_3 \xi_3 = \chi_{12}|_1 (\xi_1 + \lambda_{12}\xi_3) + \chi_{12}|_2 (\xi_2 + \lambda_{21}\xi_3). \quad (97)$$

Proceeding analogously for the other two sides we construct the matrix relation

$$\begin{Bmatrix} \chi_{12} \\ \chi_{23} \\ \chi_{31} \end{Bmatrix} = \begin{bmatrix} \xi_1 + \lambda_{12}\xi_3 & \xi_2 + \lambda_{21}\xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_2 + \lambda_{23}\xi_1 & \xi_3 + \lambda_{32}\xi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_3 + \lambda_{31}\xi_2 & \xi_1 + \lambda_{13}\xi_2 \end{bmatrix} \mathbf{g}, \quad (98)$$

or

$$\chi = \mathbf{A}_\chi \mathbf{g}, \quad \kappa = \mathbf{T} \mathbf{A}_\chi \mathbf{g}. \quad (99)$$

Since  $\mathbf{T}$  is constant we can do Step 3 of Table 3 directly on the natural curvatures. Now  $\mathbf{A}_\chi(\xi_1, \xi_2, \xi_3)$  is a linear function of the triangular coordinates. Consequently, the mean natural curvatures can be simply obtained by evaluating  $\mathbf{A}_\chi$  at the centroid  $\xi_1 = \xi_2 = \xi_3 = 1/3$ . Let the corresponding matrix be  $\bar{\mathbf{A}}_\chi$ . Then  $\bar{\chi} = \bar{\mathbf{A}}_\chi \mathbf{g}$ , and the natural deviatoric curvatures are given by

$$\chi_d = (\mathbf{A}_\chi - \bar{\mathbf{A}}_\chi) \mathbf{g}, \quad (100)$$

which transformed to deviatoric Cartesian curvatures  $\kappa_d = \kappa - \bar{\kappa}$  gives finally

$$\kappa_d = \mathbf{T}(\mathbf{A}_\chi - \bar{\mathbf{A}}_\chi) \mathbf{g} = \mathbf{A}_d \mathbf{g}. \quad (101)$$

### 11.6 The Element Stiffness Matrix

The basic stiffness matrix  $\mathbf{K}_b$  is the same derived in Ref. 8 using the conventional FF and need not be rederived here. The higher order stiffness matrix is given by Eqs. 84, which for a plate bending element specialise to

$$\mathbf{K}_h = \alpha \mathbf{Q}^T \mathbf{K}_{ad} \mathbf{Q} = \alpha \mathbf{Q}^T \left[ \int_A \mathbf{A}_d^T \mathbf{D} \mathbf{A}_d dA \right] \mathbf{Q}, \quad (102)$$

where  $\mathbf{D}$  is the Cartesian moment-curvature constitutive matrix resulting from the integration of  $\mathbf{E}$  through the plate thickness:

$$\mathbf{m} = \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \mathbf{D} \kappa. \quad (103)$$

Since  $\mathbf{A}_d$  varies linearly, if  $\mathbf{D}$  is constant we could numerically integrate  $\mathbf{K}_{ad}$  in Eq. 102 exactly with a three point Gauss rule, for example the three midpoint formula. But as the formation of the element stiffness is dominated by these calculations it is of interest to derive  $\mathbf{K}_{ad}$  in closed form. Such a derivation is provided in Ref. 17.

### 11.7 Preliminary Evaluation

As of this writing only a sketchy evaluation of the first ANDES element is available. We have found that for triangles with good aspect ratio its behavior is similar to that of the scaled FF element of Ref. 8, which is known to be an excellent performer. But the ANDES element shows less distortion sensitivity for high aspect ratio elements, as can be expected from its construction. Additional evaluation details will be reported in Ref. 17.

These preliminary results are encouraging in that we now have two good stand-alone components (FF and ANDES) of  $\mathbf{K}_h$ . Thus it is plausible that a weighted mix of these formulations as per Eq. 72 can be used to squeeze the ultimate in performance for this very simple element.

## 12. CONCLUSIONS

The results presented in this paper may be summarized as follows.

1. The classical variational principles of linear elasticity may be embedded in a parametrised matrix form.
2. The elasticity principles with independently varied displacements are members of a three-parameter family. Those principles without independent displacements are members of a one-parameter family.
3. Finite element assumptions for constructing high-performance elements may be conveniently investigated on this family using hybrid forcing potentials.
4. Kinematic constraints established outside the realm of the variational principle may be incorporated through Lagrange multiplier adjunction.
5. The FF and ANS methods for constructing HP finite elements may be presented within this augmented variational setting. A variant of ANS, called ANDES, fits naturally the decomposition of the stiffness equations into basic and higher order parts. In addition, combined FF/ANDES forms emerge from the general parametrized principle.
6. The satisfaction of the individual element test yields various orthogonality conditions that the kinematic constraints should satisfy *a priori*.
7. The first ANDES element based on this formulation displays encouraging stand-alone performance as regards distortion sensitivity. The weighted combination of this element with its FF counterpart remains a topic for further investigation.

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13. ABSTRACT (Maximum 200 words) High performance elements are simple finite elements constructed to deliver engineering accuracy with coarse arbitrary grids. This paper is part of a series on the variational foundations of high-performance elements, with emphasis on plate and shell elements constructed with the free formulation (FF) and assumed natural strain (ANS) methods. In the present paper we study parametrized variational principles that provide a common foundation for the FF and ANS methods, as well as for a combination of both. From this unified formulation a variant of the ANS formulation, called the assumed natural deviatoric strain (ANDES) formulation, emerges as an important special case. The first ANDES element: a high-performance 9-dof triangular Kirchhoff plate bending element, is briefly described to illustrate the use of the new formulation.				
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